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From the Liouville Equation to the Generalized Boltzmann Equation for Magnetotransport in the 2D Lorentz Model

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We consider a system of non-interacting charged particles moving in two dimensions among fixed hard scatterers, and acted upon by a perpendicular magnetic field. Recollisions between charged particles and scatterers are unavoidable in this case. We derive from the Liouville equation for this system a generalized Boltzmann equation with infinitely long memory, but which still is analytically solvable. This kinetic equation has been earlier written down from intuitive arguments.

KEY WORDS: Kinetic theory; non-Markovian effects; magnetotransport; Lorentz model.

1. INTRODUCTION

The Lorentz model[1–3] where a particle moves among and collides with fixed scatterers has provided a rich testing ground for kinetic theory. In particular, the Boltzmann equation is not only exactly solvable in this model, but the equation itself is exact in the Grad limit (to be defined below). The *Stosszahlansatz* used in constructing the Boltzmann equation

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relies implicitly on the assumption that the particle never returns to a scatterer after having collided with it. The probability of such recollisions vanishes in the Grad limit. On this basis the Boltzmann equation for the standard Lorentz model is taken to be exact in the Grad limit.

Recently, however, an interesting exception was discovered[4, 5] to this state of affairs. Let the particle have electric charge $-e$ and move on a plane pierced by a perpendicular constant magnetic field B. The particle moves along circle arcs between collisions, and if it does not encounter a new scatterer along the arc, it will *recollide* with the initial scatterer. This destroys the above assumption and renders the Boltzmann equation invalid. Note that both the (two-) dimensionality of space and the presence of a perpendicular magnetic field are essential here.

In refs. 4 and 5 a generalization of the conventional Boltzmann equation was proposed that takes into account consecutive recollisions with the same scatterer. The arguments leading to this equation were intuitive, on the same level as the *Stosszahlansatz* itself. Numerical simulations[6] support the results that were found using this generalized Boltzmann equation. The aim of the present paper is to provide a microscopic underpinning to the generalized Boltzmann equation in the form of a systematic derivation from the Liouville equation. As a background we start by summarizing the intuitive arguments leading to the generalized Boltzmann equation.

The charged particle moves on a plane of (large) area A, with N randomly placed hard disk scatterers of radius a. We denote by $n=N/A$ their number density. The disks do not overlap.

The generalized Boltzmann equation describes the evolution of the probability density $f_1(\mathbf{x}, \mathbf{v}, t)$ for finding the moving particle at time t at position $\mathbf{x}$ with velocity $\mathbf{v}$. This non-markovian kinetic equation has the form

$$\frac{D}{Dt} f^G(\mathbf{x}, \mathbf{v}, t) = na \sum_{k=0}^{[t/T_0]} e^{-\nu k T_0} \int_{S^1} d\mathbf{n}(\mathbf{v}\cdot\mathbf{n}) \times [\theta(\mathbf{v}\cdot\mathbf{n})\, b_{\mathbf{n}} + \theta(-\mathbf{v}\cdot\mathbf{n})]\, f^G(\mathbf{x}, S_0^{-k}\mathbf{v}, t - kT_0) \tag{1}$$

where

$$f^G(\mathbf{x}, \mathbf{v}, t) = \begin{cases} f_1(\mathbf{x}, \mathbf{v}, t) & \text{if} \quad 0 < t < T_0 \\ (1 - e^{-\nu T_0})\, f_1(\mathbf{x}, \mathbf{v}, t) & \text{if} \quad t > T_0 \end{cases} \tag{2}$$

and where $\nu = 2\,|\mathbf{v}|\, na$ is the collision frequency and T_0 the cyclotron period. Furthermore,

$$\frac{D}{Dt} = \left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + (\boldsymbol{\omega}\times\mathbf{v})\cdot\frac{\partial}{\partial \mathbf{v}}\right] \tag{3}$$

is the generator of free cyclotron motion with frequency $|\boldsymbol{\omega}| = \omega = eB/m$, and $[t/T_0]$ the number of cyclotron periods $T_0 = 2\pi/\omega$ completed before time t. The angular integration over the unit vector $\mathbf{n}$ in (1) is over the entire unit sphere S^1 centered at the origin. In the gain term (positive contribution), there appears the distribution acted upon by the operator $b_{\mathbf{n}}$, defined by

$$b_{\mathbf{n}}\phi(\mathbf{v}) = \phi(\mathbf{v} - 2(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}) \tag{4}$$

where ϕ is an arbitrary function of $\mathbf{v}$. The precollisional velocity $\mathbf{v}' = \mathbf{v} - 2(\mathbf{v}\cdot\mathbf{n})\,\mathbf{n}$ becomes $\mathbf{v}$ after the elastic collision with the immobile (infinitely massive) scattering disk. Note that $\mathbf{v}'\cdot\mathbf{n} < 0$. In the loss term (negative contribution), the precollisional velocity, $\mathbf{v}$, is also from the hemisphere $\mathbf{v}\cdot\mathbf{n} < 0$. Finally, the shift operator S_0^{-k}, when acting on $\mathbf{v}$, rotates the velocity through the angle $-k\psi$, where ψ is the scattering angle (from $\mathbf{v}'$ to $\mathbf{v}$).

When $0 < t < T_0$, $[t/T_0] = 0$, and $f^G(\mathbf{x}, \mathbf{v}, t) = f_1(\mathbf{x}, \mathbf{v}, t)$. No recollisions are yet possible and Eq. (1) reduces to the standard Boltzmann equation

$$\begin{aligned}\frac{D}{Dt} f_1(\mathbf{x}, \mathbf{v}, t) &= na \int_{S^1} d\mathbf{n}(\mathbf{v}\cdot\mathbf{n})[\theta(\mathbf{v}\cdot\mathbf{n})\, b_{\mathbf{n}} + \theta(-\mathbf{v}\cdot\mathbf{n})]\, f_1(\mathbf{x}, \mathbf{v}, t) \\ &= \mathscr{C}^B f_1(\mathbf{x}, \mathbf{v}, t)\end{aligned} \tag{5}$$

where $\mathscr{C}^B$ is the Boltzmann collision operator.

When $t > T_0$, the distribution $f_1(\mathbf{x}, \mathbf{v}, t)$ splits into two parts. With probability $\exp(-\nu T_0)$ the charged particle continues to perform free cyclotron motion, having explored the whole circle during the first period T_0. With probability $[1 - \exp(-\nu T_0)]$ the particle suffers collisions and becomes a wandering particle among the hard disks. Since the probability of being a wandering particle is less than unity, the distribution for $t > T_0$ should be renormalized. Hence the need for (2).

The sum in the generalized equation (1), from $k = 1$ to $k = [t/T_0]$, takes into account all possible recollision events. The collision "now" can be the kth recollision, with every recollision having the same scattering angle ψ, equal to the one of the initial collision at $t - kT_0$. Only this initial collision, with incoming velocity $S_0^{-k}\mathbf{v}$, is described by the Boltzmann operator $\mathscr{C}^B$, all subsequent ones follow from dynamics alone, weighted by the survival probability from one collision to the next, $\exp(-\nu T_0)$. Thus, for a sequence of k recollisions one gets the factor $\exp(-k\nu T_0)$. Finally, summation over k from 1 to $[t/T_0]$ takes all possible recollision sequences into account.

On the basis of an intuitive derivation, as sketched above, and by analogy to the results for the general Lorentz model, refs. 4 and 5 assume that the generalized Boltzmann equation (1) gives an exact description of the time evolution of our system in the Grad limit,

$$\lim_{\text{Grad}} = \begin{cases} a \to 0 \text{ and } N/A = n \to \infty \\ na = \text{const} \end{cases} \tag{6}$$

According to (6) the radius of the scattering disks approaches zero but their number density increases at the same time in such a way that the mean free path of the wandering particle, $\Lambda = (2na)^{-1}$ remains constant.

In the next section, we discuss the initial value problem for the Liouville equation. In Section 3, the effect of recollisions is analyzed, and Section 4 contains the derivation of the generalized Boltzmann equation. We conclude with a short summary. The shift operator along the trajectory involving recollisions is described in the appendix.

2. DYNAMICAL EQUATIONS

We denote by $F(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1,..., \mathbf{y}_N)$ the joint probability density for finding the moving point charge at time t at point $\mathbf{x}$ with velocity $\mathbf{v}$ among N hard disk scatterers located at positions $\mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N$. F satisfies the normalization condition

$$\int_{\mathbf{R}^2} d\mathbf{v} \int_{\mathbf{\Omega}} d\mathbf{x} \int_{\Omega^N} d\mathbf{y}_1 \cdots d\mathbf{y}_N \, F(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N) = 1 \tag{7}$$

and is the solution of the Liouville equation

$$\frac{D}{Dt} F(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N) = a \sum_{j=1}^{N} T(\mathbf{x} - \mathbf{y}_j, \mathbf{v}) \, F(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1, \mathbf{y}_2,..., \mathbf{y}_N) \tag{8}$$

where $T(\mathbf{x} - \mathbf{y}_j, \mathbf{v})$ is the binary collision operator for scatterer j,

$$T(\mathbf{x} - \mathbf{y}_j, \mathbf{v}) = \int_{S^1} d\mathbf{n} (\mathbf{v} \cdot \mathbf{n}) [\theta(\mathbf{v} \cdot \mathbf{n}) \, b_{\mathbf{n}} + \theta(-\mathbf{v} \cdot \mathbf{n})] \, \delta(\mathbf{x} - \mathbf{y}_j - a\mathbf{n}) \tag{9}$$

The operator $b_{\mathbf{n}}$ acting on the velocity vector has been defined in (4), and $a\mathbf{n}$ is the point of impact with respect to the center of the disk.

At time $t=0$, the initial condition for the Liouville equation (8) is assumed to be of the form

$$F(\mathbf{x}, \mathbf{v}, t=0; \mathbf{y}_1,..., \mathbf{y}_N) = f_1(\mathbf{x}, \mathbf{v}, 0) \left[\prod_{i=1}^{N} \theta(|\mathbf{x}-\mathbf{y}_i| - a) \right] \frac{\rho(\mathbf{y}_1,..., \mathbf{y}_N)}{1-\pi n a^2} \tag{10}$$

Here $f_1(\mathbf{x}, \mathbf{v}, 0)$ is the initial state of the particle, and $\rho(\mathbf{y}_1,..., \mathbf{y}_N)$ describes the static, non-overlapping distribution of scattering disks,

$$\rho(\mathbf{y}_1,..., \mathbf{y}_N) = \prod_{1 \leqslant i < j \leqslant N} \theta(|\mathbf{y}_i - \mathbf{y}_j| - 2a)/\mathcal{N} \tag{11}$$

where $\mathcal{N}$ is the normalizing factor such that $\int_{\Omega^N} d\mathbf{y}_1 \cdots \mathbf{y}_N \rho(\mathbf{y}_1,..., \mathbf{y}_N) = 1$.

To complete the statement of the problem, we shall for simplicity assume that the region Ω enclosing the system has finite area A and no boundaries (e.g., a two-dimensional torus). We furthermore assume that the linear dimensions of Ω are very large compared to the cyclotron radius.

The factor $[1-\pi n a^2]^{-1}$ assures the proper normalization (7) of the initial condition since

$$\int_{\Omega^N} d\mathbf{y}_1 \cdots d\mathbf{y}_N \, \rho(\mathbf{y}_1,..., \mathbf{y}_N) \left[\prod_{i=1}^{N} \theta(|\mathbf{x}-\mathbf{y}_i| - a) \right] = 1 - \pi n a^2 \tag{12}$$

Clearly, the probability density $f_1(\mathbf{x}, \mathbf{v}, t)$ is obtained from $F(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1,..., \mathbf{y}_N)$ by integration over the positions of all scatterers,

$$f_1(\mathbf{x}, \mathbf{v}, t) = \int_{\Omega^N} d\mathbf{y}_1 \cdots d\mathbf{y}_N \, F(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1,..., \mathbf{y}_N) \tag{13}$$

Integrating the Liouville equation (8) we thus find

$$\frac{D}{Dt} f_1(\mathbf{x}, \mathbf{v}, t) = aN \int_{\Omega} d\mathbf{y}_1 \, T(\mathbf{x}-\mathbf{y}_1, \mathbf{v}) \, f_2(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1) \tag{14}$$

where

$$f_2(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1) = \int_{\Omega^{N-1}} d\mathbf{y}_2 \cdots d\mathbf{y}_N \, F(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1,..., \mathbf{y}_N) \tag{15}$$

Equation (14) will be the main object of our study. Using the explicit form of the collision operator (9), we rewrite it as

$$\frac{D}{Dt} f_1(\mathbf{x}, \mathbf{v}, t) = an \int_{S^1} d\mathbf{n} (\mathbf{v} \cdot \mathbf{n}) [\theta(\mathbf{v} \cdot \mathbf{n}) \, b_{\mathbf{n}} + \theta(-\mathbf{v} \cdot \mathbf{n})] \, I_a(\mathbf{x}, \mathbf{v}, \mathbf{n}, t) \tag{16}$$

where

$$I_a(\mathbf{x}, \mathbf{v}, \mathbf{n}, t) = A \int_{\Omega^{N-1}} d\mathbf{Y}\, F(\mathbf{x}, \mathbf{v}, t; \mathbf{x} - a\mathbf{n}, \mathbf{Y}) \tag{17}$$

where from now on we use the short-hand notation $\mathbf{Y} = (\mathbf{y}_2, ..., \mathbf{y}_N)$.

In Eq. (17), $\mathbf{y}_1 = \mathbf{x} - a\mathbf{n}$, which corresponds to the precollisional presence of the particle at the surface of the scatterer. As has already been mentioned in the Introduction, only velocities such that $(\mathbf{v} \cdot \mathbf{n}) < 0$ appear as argument in the distribution F.

Our aim here is to demonstrate that Eq. (16) becomes a closed kinetic equation for the distribution $f_1(\mathbf{x}, \mathbf{v}, t)$ in the Grad limit (6), and that taking this limit we shall recover the generalized Boltzmann equation (1).

We note that the Liouville equation (8) implies the relation

$$F(\mathbf{x} = \mathbf{y} + a\mathbf{n}, \mathbf{v}, t; \mathbf{y}, \mathbf{Y}) = F(\mathbf{y} + aS_{-t}^{(N)}\mathbf{n}, S_{-t}^{(N)}\mathbf{v}, t = 0; \mathbf{y}, \mathbf{Y}) \tag{18}$$

where $S_{-t}^{(N)}$ denotes the backwards shift operator along the exact phase trajectory of the N-scatterer problem. A crucial point is to separate clearly two possibilities: (a) The particle did not collide with the scatterer at $\mathbf{y}$ before time t. (b) It did (leading to the problem of recollisions). The consequences of this distinction will be discussed in the next section.

3. RECOLLISIONS

For any fixed configuration of scatterers $(\mathbf{y}, \mathbf{Y}) = (\mathbf{y}, \mathbf{y}_2, ..., \mathbf{y}_N)$, one can construct the past history for the moving particle leading to the position $\mathbf{y} + a\mathbf{n}$ and velocity $\mathbf{v}$ at time t. The probability weight for the initial conditions at $t = 0$ is defined in Eq. (10).

At $t > 0$, when $\mathbf{x} = \mathbf{y} + a\mathbf{n}$, the particle is about to collide with the scatterer at $\mathbf{y}$, since $(\mathbf{v} \cdot \mathbf{n}) < 0$. Depending on the past history, we separate the integration domain Ω^{N-1} in (17) into a union of disjoint time-dependent subdomains $\Delta_k(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t)$ defined as

$$\Delta_0(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) = \{\mathbf{Y} : \text{no collision with scatterer at } \mathbf{y} \text{ before } t\} \tag{19}$$

and

$$\Delta_k(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) = \{\mathbf{Y} : k \text{ collision with scatterer at } \mathbf{y} \text{ before } t\} \tag{20}$$

where $k = 1, 2, ..., \infty$. Clearly, we have

$$\Delta_j(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) \cap \Delta_k(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) = \varnothing \tag{21}$$

when $j \neq k$, and

$$\bigcup_{k=0}^{\infty} \Delta_k(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) = \Omega^{N-1} \tag{22}$$

Thus, we may rewrite Eq. (17) as

$$I_a(\mathbf{y} + a\mathbf{n}, \mathbf{v}, \mathbf{n}, t) = \sum_{k=0}^{\infty} A \int_{\Delta_k(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t)} d\mathbf{Y}\, F(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t; \mathbf{y}, \mathbf{Y}) \tag{23}$$

We now introduce a further partition of each $\Delta_k(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t)$, $k = 1,..., \infty$, into two disjoint parts,

$$\Delta_k(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) = \Delta_k^{(0)}(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) \cup \Delta_k^{(1)}(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) \tag{24}$$

$$\Delta_k^{(0)}(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) \cap \Delta_k^{(1)}(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) = \varnothing \tag{25}$$

such that

$$\Delta_k^{(0)}(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) = \{\mathbf{Y} : \text{no collisions with scatterers at } \mathbf{Y} \text{ within the time interval } [t - kT_a, t]\} \tag{26}$$

Here T_a is the period between two successive collisions of the particle with the scatterer at $\mathbf{y}$.

If $\mathbf{Y} \in \Delta_k^{(0)}(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t)$, then the particle collided with the scatterer at $\mathbf{y}$ k times during the time interval $[t - kT_a, t]$. Furthermore, it had not collided with this scatterer before $t - kT_a$. We may therefore conclude that

$$\mathbf{Y} \in \Delta_k^{(0)}(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t) \Rightarrow F(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t; \mathbf{y}, \mathbf{Y}) = F(\mathbf{y} + aS_a^{-k}\mathbf{n}, S_a^{-k}\mathbf{v}, t - kT_a; \mathbf{y}, \mathbf{Y}) \tag{27}$$

where S_a is the shift operator along the trajectory of the particle between two collisions with the sactterer at $\mathbf{y}$, in the absence of other scatterers. We construct this operator explicitly in the Appendix, see also ref. 5.

In the Grad limit, the volume of $\mathbf{Y}$-space for which collisions with other scatterers occur between recollisions with the same scatterer is negligible compared to the volume corresponding to a sequence of consecutive recollisions, and in fact

$$\lim_{\text{Grad}} \|\Delta_k^{(1)}\| = 0 \tag{28}$$

where $\| \cdots \|$ represents the relative volume with respect to the set Ω^{N-1}. The reason behind Eq. (28) is that the probability to return to the scatterer at $\mathbf{y}$ after an intermediate collision with another scatterer involves an extra power of a and thus vanishes in the Grad limit. The corresponding problem in the absence of a magnetic field is discussed in ref. 2. Close to the Grad limit, (28) allows us to change the volume of integration in (23) from $\Delta_k(\mathbf{y}+a\mathbf{n}, \mathbf{v}, t)$ to $\Delta_k^{(0)}(\mathbf{y}+a\mathbf{n}, \mathbf{v}, t)$.

Let us introduce the limit function

$$\mathscr{F}_k(\mathbf{y}, S_0^{-k}\mathbf{v}, t-kT_0) = \lim_{\text{Grad}} \left(\frac{A}{\|\Delta_k^{(0)}(\mathbf{y}+a\mathbf{n}, \mathbf{v}, t)\|} \times \int_{\Delta_k^{(k)}(\mathbf{y}+a\mathbf{n}, \mathbf{v}, t)} d\mathbf{Y}\, F(\mathbf{y}+a\mathbf{n}, \mathbf{v}, t; \mathbf{y}, \mathbf{Y}) \right); \qquad \Delta_0^{(0)} \equiv \Delta_0 \tag{29}$$

where (27) has been used for convenience. This limit function no longer depends on the number of scatterers, N.

In terms of $\mathscr{F}_k$, the Grad limit of (23) becomes

$$I_0(\mathbf{y}, \mathbf{v}, \mathbf{n}, t) = \sum_{k=0}^{[t/T_0]} \|\Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t)\| \, \mathscr{F}_k(\mathbf{y}, S_0^{-k}\mathbf{v}, t-kT_0) \tag{30}$$

Note, with reference to the Appendix, that the shift operator S_0^{-k} depends on the vector $\mathbf{n}$, even in the Grad limit.

The relation

$$\Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t) = \Delta_0(\mathbf{y}, S_0^{-k}\mathbf{v}, t-kT_0) \cap \Gamma(t-kT_0, t) \tag{31}$$

is valid in the Grad limit where the set $\Gamma(t_1, t_2)$ corresponds to those regions of $\mathbf{Y}$-space in which the particle suffers no collisions with the $N-1$ scatterers at $\mathbf{Y}$ during the time interval (t_1, t_2).

We now evaluate the volumes of the sets $\Delta_0(\mathbf{y}, S_0^{-k}\mathbf{v}, t-kT_0)$ and $\Gamma(t_1, t_2)$. In order to do so, we define the set complementary to $\Delta_0(\mathbf{y}, \mathbf{v}, t)$,

$$\bar{\Delta}_0(\mathbf{y}, \mathbf{v}, t) = \{\mathbf{Y} : \text{at least one collision with scatterer at } \mathbf{y} \text{ before } t\} \tag{32}$$

If $t \geqslant T_0$, two types of histories consistent with the condition in (32) are possible: (a) No collisions with other scatterers occured within the time interval $(t-T_0, t)$. (b) The particle collided with other scatterers after the previous collision with the scatterer at $\mathbf{y}$. Close to the Grad limit possibility (b) is improbable and in the limit, possibility (a) is realized with probability 1. If, on the other hand, $t < T_0$, the condition in (32) can only be realized by possibility (b), the probability of which vanishes in the Grad limit. Thus, the characteristic function χ is, in this limit,

$$\chi_{\bar{\Delta}_0(\mathbf{y},\,\mathbf{v},\,t)} = \begin{cases} 0 & \text{if} \quad 0 < t < T_0 \\ \chi_{\Gamma(t - T_0,\, t)} & \text{if} \quad t > T_0 \end{cases} \tag{33}$$

The characteristic function for the set $\Delta_0(\mathbf{y}, \mathbf{v}, t)$ is

$$\chi_{\Delta_0(\mathbf{y},\,\mathbf{v},\,t)} = 1 - \chi_{\bar{\Delta}_0(\mathbf{y},\,\mathbf{v},\,t)} \tag{34}$$

We now evaluate the characteristic function of the set $\Gamma(t_1, t_2)$. Let $\mathscr{L}$ be the trajectory of the particle for $t_1 < \tau < t_2$. Then,

$$\mathscr{L} = \{\mathbf{x}(\tau) \mid t_1 < \tau < t_2\} \tag{35}$$

The distance between $\mathscr{L}$ and a given point $\mathbf{a}$ is

$$\operatorname{dist}(\mathscr{L}, \mathbf{a}) = \min_{t_1 < \tau < t_2} |\mathbf{x}(\tau) - \mathbf{a}| \tag{36}$$

The characteristic function $\chi_{\Gamma(t_1,\, t_2)}$—for finite N—is given by

$$\chi_{\Gamma(t_1,\, t_2)} = \prod_{j=2}^{N} \theta[\operatorname{dist}(\mathscr{L}, \mathbf{y}_j) - a] \tag{37}$$

The volume of the set $\Gamma(t_1, t_2)$ is then the integral

$$\|\Gamma(t_1, t_2)\| = \frac{1}{A^{N-1}} \int_{\Omega^{N-1}} d\mathbf{Y}\, \chi_{\Gamma(t_1,\, t_2)} \approx \left[\frac{1}{A} \int_{\Omega} d\mathbf{y}\, \theta(\operatorname{dist}(\mathscr{L}, \mathbf{y}) - a) \right]^{N-1} \tag{38}$$

We have on the right-hand side of this expression assumed that we are close to the Grad limit so that the positions of the scatterers become independent of each other. We evaluate the integral

$$\frac{1}{A} \int_{\Omega} d\mathbf{y}\, \theta(\operatorname{dist}(\mathscr{L}, \mathbf{y}) - a) = \left[1 - \frac{2aL(t_1, t_2)}{A} \right] + \mathscr{O}(a^2) \tag{39}$$

where $L(t_1, t_2)$ is the length of the curve $\mathscr{L}$. Combining (38) and (39), and going to the Grad limit, we find that

$$\|\Gamma(t_1, t_2)\| = e^{-2naL(t_1,\, t_2)} = e^{-\nu(t_2 - t_1)} \tag{40}$$

where we have used that the particle moves with constant speed v so that $L(t_1, t_2) = v(t_2 - t_1)$.

Using Eq. (40) combined with Eqs. (33) and (34), we find that

$$\|\Delta_0(\mathbf{y}, \mathbf{v}, t)\| = \begin{cases} 1 & \text{if} \quad 0 < t < T_0 \\ 1 - e^{-\nu T_0} & \text{if} \quad t > T_0 \end{cases} \tag{41}$$

We are now in the position to calculate the volume of $\Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t)$ through Eq. (31) combined with Eqs. (40) and (41).

The set $\Gamma(t - kT_0, t)$, consists of all $\mathbf{Y}$ such that the particle has k consecutive collisions with the scatterer at $\mathbf{y}$ without collisions with other scatterers. If we follow the motion of the particle during the time interval $[t - (k+1)\, T_0, t - kT_0]$ that preceeds the time interval for which $\Gamma(t - kT_0, t)$ is defined, and we assume t large enough so that $t - (k+1)\, T_0 > 0$, two histories are possible: (a) The particle has collided with other scatterers within the time interval $[t - (k+1)\, T_0, t - kT_0]$, or (b) it has collided with the scatterer at $\mathbf{y}$, in which case it has *not* collided with any other scatterer during this period. If (a) is the case, then the configuration belongs to the set $\Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t)$—see Eq. (31)—otherwise the configuration belongs to the set $\Gamma(t - (k+1)\, T_0, t)$. Thus, we have that

$$\Gamma(t - kT_0, t) = \Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t) \cup \Gamma(t - (k+1)\, T_0, t) \tag{42}$$

Furthermore,

$$\Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t) \cap \Gamma(t - (k+1)\, T_0, t) = \varnothing \tag{43}$$

The volume of the set $\Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t)$ is therefore

$$\|\Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t)\| = \|\Gamma(t - kT_0, t)\| - \|\Gamma(t - (k+1)\, T_0, t)\| \tag{44}$$

Equation (40), which is exact in the Grad limit, gives

$$\|\Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t)\| = e^{-\nu k T_0}(1 - e^{-\nu T_0}) \qquad \text{if} \quad t > (k+1)\, T_0 \tag{45}$$

If we now assume that $kT_0 < t < (k+1)\, T_0$ so that $\Gamma(t - (k+1)\, T_0, t) = \varnothing$, we have that $\Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t) = \Gamma(t - kT_0, t)$ and consequently,

$$\|\Delta_k^{(0)}(\mathbf{y}, \mathbf{v}, t)\| = e^{-\nu k T_0} \qquad \text{if} \quad kT_0 < t < (k+1)\, T_0 \tag{46}$$

Using Eqs. (45) and (46) (remember that S_0^{-k} depends on $\mathbf{n}$), we may rewrite Eq. (30)

$$I_0(\mathbf{y}, \mathbf{v}, \mathbf{n}, t) = (1 - e^{-\nu T_0}) \sum_{k=0}^{[t/T_0]-1} e^{-\nu k T_0} \mathscr{F}_k(\mathbf{y}, S_0^{-k}\mathbf{v}, t - kT_0) + e^{-\nu [t/T_0] T_0} \mathscr{F}_{[t/T_0]}\left(\mathbf{y}, S_0^{-[t/T_0]}\mathbf{v}, t - \left[\frac{t}{T_0}\right] T\right) \tag{47}$$

Our next task is to find the relation between $f_1(\mathbf{x}, \mathbf{v}, t)$ and $\mathscr{F}_k(\mathbf{y}, \mathbf{v}, t)$.

4. THE GENERALIZED BOLTZMANN EQUATION

The Grad limit of Eq. (16) becomes

$$\frac{D}{Dt} f_1(\mathbf{x}, \mathbf{v}, t) = an \int_{S^1} d\mathbf{n} (\mathbf{v} \cdot \mathbf{n}) [\theta(\mathbf{v} \cdot \mathbf{n})\, b_{\mathbf{n}} + \theta(-\mathbf{v} \cdot \mathbf{n})]\, I_0(\mathbf{x}, \mathbf{v}, \mathbf{n}, t) \tag{48}$$

with I_0 given by Eq. (47) with $\mathbf{y} = \mathbf{x}$.

If $t < T_0$, as a consequence of Eq. (47) $I_0(\mathbf{x}, \mathbf{v}, \mathbf{n}, t) = \mathscr{F}_0(\mathbf{x}, \mathbf{v}, t)$. To express $\mathscr{F}_0(\mathbf{x}, \mathbf{v}, t)$ through $f_1(\mathbf{x}, \mathbf{v}, t)$, we consider the right hand side of (29) with $k = 0$ and set for the moment

$$F(\mathbf{x}, \mathbf{v}, t; \mathbf{y}, \mathbf{Y}) \equiv F^{(N)}(\mathbf{x}, \mathbf{v}, t; \mathbf{y}, \mathbf{Y}) \tag{49}$$

to stress that it is a solution of the N-scatterer problem. Near the Grad limit, i.e., for sufficiently large N and small a provided $Na = \text{constant}$, we have

$$F^{(N)}(\mathbf{x}, \mathbf{v}, t=0; \mathbf{y}, \mathbf{Y}) \approx \frac{1}{A} F^{(N-1)}(\mathbf{x}, \mathbf{v}, t=0; \mathbf{Y}) \tag{50}$$

(See Eq. (10) for $a \to 0$.) Assuming that $\mathbf{Y} \in \Delta_0(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t)$ (i.e., no collisions with the scatterer at $\mathbf{y}$ before time t), we obtain from (18)

$$F^{(N)}(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t; \mathbf{y}, \mathbf{Y}) = F^{(N)}(\mathbf{y} + aS_{-t}^{(N)}\mathbf{n}, S_{-t}^{(N-1)}\mathbf{v}, t=0; \mathbf{y}, \mathbf{Y}) \tag{51}$$

Combining the last two equalities, we get

$$F^{(N)}(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t; \mathbf{y}, \mathbf{Y}) \approx \frac{1}{A} F^{(N-1)}(\mathbf{y} + a\mathbf{n}, \mathbf{v}, t; \mathbf{Y}) \tag{52}$$

for $\mathbf{Y} \in \Delta_0(\mathbf{y}+a\mathbf{n}, \mathbf{v}, t)$. Thus, near the Grad limit we have from Eq. (29)

$$\mathscr{F}_0(\mathbf{y}, \mathbf{v}, t) \approx \frac{1}{\|\Delta_0(\mathbf{y}+a\mathbf{n}, \mathbf{v}, t)\|} \int_{\Delta_0(\mathbf{y}+a\mathbf{n}, \mathbf{v}, t)} d\mathbf{Y}\, F^{(N-1)}(\mathbf{y}+a\mathbf{n}, \mathbf{v}, t; \mathbf{Y}) \tag{53}$$

for all $t > 0$. The simplest case $0 < t < T_0$, leads to the approximate equalities

$$\Delta_0(\mathbf{y}, \mathbf{v}, t) \approx \Omega^{N-1} \tag{54}$$

$$\|\Delta_0(\mathbf{y}, \mathbf{v}, t)\| \approx 1 \Rightarrow \mathscr{F}_0(\mathbf{y}, \mathbf{v}, t) \approx f_1(\mathbf{y}, \mathbf{v}, t) \tag{55}$$

which become exact in the Grad limit—see Eq. (41). Hence, for $0 < t < T_0$, Eq. (48) is identical to the usual Boltzmann equation (5) for $f_1(\mathbf{x}, \mathbf{v}, t)$. We remark that this is perhaps the simplest way to extract the Boltzmann equation directly from the Liouville equations in the Grad limit.

When $t > T_0$, Ω^N—which is the space over which $F^{(N)}$ is averaged in order to obtain f_1—is split into two disjoint sets,

$$\Omega^N = \mathscr{A}_0 \cup \mathscr{A}_1 \tag{56}$$

where

$$\mathscr{A}_0 = \{(\mathbf{y}_1,..., \mathbf{y}_N) : \text{no collisions during the period } (t - T_0, t)\} \tag{57}$$

Both subsets $\mathscr{A}_0$ and $\mathscr{A}_1$ are defined for any given phase-time point $(\mathbf{x}, \mathbf{v}, t)$, $t > T_0$. If the particle is not scattered during the time $(t - T_0, t)$, there was no earlier scattering due to the periodicity of the motion. Thus, the subsets $\mathscr{A}_0$ and $\mathscr{A}_1$ do not actually depend on time $t > T_0$ and can be equally well defined at $t = T_0$. The solution of the Liouville equation (8) with initial conditions (10) for $(\mathbf{y}_1, \mathbf{Y}) \in \mathscr{A}_0$ is

$$F^{(N)}(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1,..., \mathbf{y}_N) = f_1(S_0(-t)\,\mathbf{x}, S_0(-t)\,\mathbf{v}, 0) \left[\prod_{i=1}^{N} \theta(|S_0(-t)\,\mathbf{x} - \mathbf{y}_i| - a) \right] \frac{\rho(\mathbf{y}_1,..., \mathbf{y}_N)}{1 - \pi n a^2} \tag{58}$$

where $S_0(\Delta t)$ is the shift operator along the collisionless trajectory of the particle from t to $t + \Delta t$. Using Eq. (40), we find that in the Grad limit

$$\|\mathscr{A}_0\| = e^{-\nu T_0} \tag{59}$$

and consequently

$$\|\mathscr{A}_1\| = 1 - e^{-\nu T_0} \tag{60}$$

We may write the distribution $f_1(\mathbf{x}, \mathbf{v}, t)$ for $t > T_0$ as the sum of F averaged over $\mathscr{A}_0$ and $\mathscr{A}_1$,

$$f_1(\mathbf{x}, \mathbf{v}, t) = \|\mathscr{A}_0\|\, f_1^{(0)}(\mathbf{x}, \mathbf{v}, t) + \|\mathscr{A}_1\|\, f_1^{(1)}(\mathbf{x}, \mathbf{v}, t) \tag{61}$$

where

$$f_1^{(i)}(\mathbf{x}, \mathbf{v}, t) = \frac{1}{\|\mathscr{A}_i\|} \int_{\mathscr{A}_i} d\mathbf{y}\, d\mathbf{Y}\, F^{(N)}(\mathbf{x}, \mathbf{v}, t; \mathbf{y}, \mathbf{Y}), \qquad i = 0, 1 \tag{62}$$

We generalize the notation (61) to $0 < t < T_0$ by setting $\mathscr{A}_0 = \varnothing$ and $\mathscr{A}_1 = \Omega^N$ in such a case.

Let us consider Eqs. (27), (29), (48) and (47) for $t > T_0$. If $k \geqslant 1$, in (29), we may repeat the argumentation leading to equality (53) and obtain

$$\begin{aligned} \mathscr{F}_k(\mathbf{y}, S_0^{-k}\mathbf{v}, t - kT_0) \approx{}& \frac{1}{\|\varDelta_k^{(0)}(\mathbf{y} + aS_0^{-k}\mathbf{n}, \mathbf{v}, t)\|} \int_{\varDelta_k^{(0)}(\mathbf{y} + aS_0^{-k}\mathbf{n}, \mathbf{v}, t)} d\mathbf{Y} \\ & \times F^{(N-1)}(\mathbf{y} + aS_0^{-k}\mathbf{n}, S_0^{-k}\mathbf{v}, t - kT_0; \mathbf{Y}) \end{aligned} \tag{63}$$

where $k = 0,..., [t/T_0]$, near the Grad limit. A minor change of notation (set $N = \tilde{N} + 1$ and $\mathbf{y}_i = \tilde{\mathbf{y}}_{i-1}$, $i = 2,...$) leads to formulas similar to the definition (62) of $f_1^{(1)}$ with the only difference that the function F is averaged not over the whole "ergodic" domain $\mathscr{A}_1$, but over a subdomain $\varDelta_k^{(0)} \subset \mathscr{A}_1$. The final step is to assume, without mathematical proof, that in the Grad limit

$$\mathscr{F}_k(\mathbf{y}, \mathbf{v}, t) = f_1^{(1)}(\mathbf{y}, \mathbf{v}, t), \qquad k = 0, 1,..., \quad t > 0 \tag{64}$$

This assumption means that we can neglect in this limit the difference between average values of $F(\mathbf{x}, \mathbf{v}, t; \mathbf{y}_1,..., \mathbf{y}_N)$ taken over different subsets $\varDelta_k^{(0)}$ of the "ergodic" set $\mathscr{A}_1$. This assumption is in fact one of the central postulates of kinetic theory (the average behavior of the test particle is the same for almost all configurations of scatterers) and we admit it here without proof. The corresponding rigorous result for the linear Boltzmann equation (without magnetic field) was obtained in ref. 7. A rigorous proof of this assumption represents a mathematical challenge beyond the scope of the present paper. For the linear Boltzmann equation *without* magnetic field, this challenge was met in ref. 7, and some of the rigorous mathematical arguments found there ca, indeed, be used to justify (64). Finally, we note that the representation of $f_1(\mathbf{x}, \mathbf{v}, t)$ in the form (61) (cycling and wandering particles, see refs. 4 and 5) makes sense only for $t > T_0$. At the first stage of the motion, $0 < t < T_0$, all particles should be considered as wandering, i.e., having a chance to undergo a collision in the future. This explains why we put $\mathscr{A}_1 = \Omega^N$ and therefore $f_1^{(1)} \equiv f_1$ for $0 < t < T_0$, see comment after Eq. (62). The above formulas (48), (58)–(62) define completely the non-markovian kinetic equation (48) in the Grad limit. The final step is to reduce the equation to a more convenient form. To this end we define f^G by Eq. (2). Thus, we have the normalization

$$\int d\mathbf{x}\, d\mathbf{v}\, f^G(\mathbf{x}, \mathbf{v}, t) = \begin{cases} 1 & \text{if} \quad 0 < t < T_0 \\ 1 - e^{-\nu T_0} & \text{if} \quad t > T_0 \end{cases} \tag{65}$$

Equation (47) may thus be written

$$I_0(\mathbf{x}, \mathbf{v}, t) = \sum_{k=0}^{[t/T_0]} e^{-\nu k T_0} f^G(\mathbf{y}, S_0^{-k}\mathbf{v}, t - kT_0) \tag{66}$$

The "non-colliding" distribution function $f_1^{(0)}$ satisfies the transport equation

$$\frac{D}{Dt} f_1^{(0)}(\mathbf{x}, \mathbf{v}, t) = 0 \tag{67}$$

The "colliding" distribution function, $f_1^{(1)}$, satisfies the equation

$$\frac{D}{Dt}(1 - e^{-\nu T_0})\, f_1^{(1)}(\mathbf{x}, \mathbf{v}, t) = n \int_{S^1} d\mathbf{n} (\mathbf{v}\cdot\mathbf{n})[\theta(\mathbf{v}\cdot\mathbf{n})\, b_{\mathbf{n}} + \theta(-\mathbf{v}\cdot\mathbf{n})] \times \left[\sum_{k=0}^{[t/T_0]} e^{-\nu k T_0} f^G(\mathbf{y}, S_0^{-k}\mathbf{v}, t - kT_0) \right] \tag{68}$$

When $t > T_0$, we use the definition (2) and write this equation

$$\frac{D}{Dt} f^G(\mathbf{x}, \mathbf{v}, t) = n \sum_{k=0}^{[t/T_0]} e^{-\nu k T_0} \int_{S^1} d\mathbf{n} (\mathbf{v}\cdot\mathbf{n})[\theta(\mathbf{v}\cdot\mathbf{n})\, b_{\mathbf{n}} + \theta(-\mathbf{v}\cdot\mathbf{n})] \times f^G(\mathbf{y}, S_0^{-k}\mathbf{v}, t - kT_0) \tag{69}$$

When $t < T_0$, we add Eqs. (67) and (68). The resulting equation has the same form as Eq. (69)—which is the generalized Boltzmann equation of refs. 4 and 5.

5. CONCLUSION

We have presented a systematic derivation of the generalized Boltzmann equation (1) from the Liouville equation in the Grad limit. The derivation was done at the traditional physicist level of rigor and we stressed several points which were not proved mathematically. Moreover, a convergence of the right hand side of Eq. (14) for $f_1(\mathbf{x}, \mathbf{v}, t)$ is obviously not equivalent to convergence of the "true" function $f_1^{(N)}$ to the solution of the generalized Boltzmann equation. Therefore, there still remains a considerable effort from a mathematical point of view to prove the validity of the generalized Boltzmann equation at the same level of rigor as it was done for the usual Boltzmann equation (see ref. 2 and references therein). However, some of the basic ideas needed and some hints at the difficulties that need to be overcome to construct such a proof can be found in the above "physical derivation."

Another open problem is to clarify the connection of the generalized Boltzmann equation with the BBGKY hierarchy. The arguments discussed above show that the validity of the generalized Boltzmann equation follows directly from N-scatterer dynamics in the Grad limit. The generalized Boltzmann equation is clearly the non-markovian equation for a system with *strong* long-time pair correlations in contradistinction to the usual Boltzmann equation. It is therefore important to understand what the correlation functions look like in this case.

In order to do this, one needs to consider the whole hierarchy and find its asymptotic solution in the Grad limit. We hope to do this in a forthcoming paper.

APPENDIX

In this Appendix, we construct the shift operator S_a which was defined in (27). It shifts the particle from a position right before a collision with the scatterer at $\mathbf{y}$ along a collisionless trajectory to a new precollisional position on the surface of the scatterer.

We consider in the following a succession of collisions between the charged particle and a single scatterer. Assume that the particle follows the trajectory marked "1" in Fig. 1. This trajectory is a circle arc centered at a distance Δ from the center of the scatterer. The scattering occurs on the surface of the disk, with a scattering angle ψ, and the particle shifts to a new cyclotron orbit "2." Clearly $-(\mathbf{v}' \cdot \mathbf{n}) = (\mathbf{v} \cdot \mathbf{n}) = |\mathbf{v}| \sin(\psi/2)$, where $\mathbf{v}'$ is the precollisional velocity and $\mathbf{v}$ the postcollisional one. As a result of this symmetry, the center of the new orbit "2" has the same distance Δ from the center of the scatterer as the previous orbit. Orbit "2" leads back to the

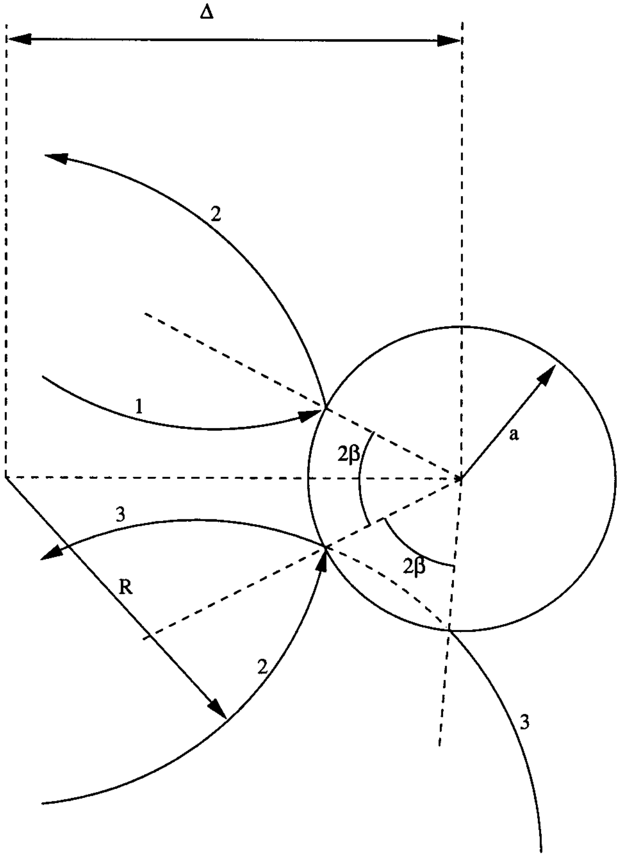

Fig. 1. The charged particle follows orbit "1" and collides with the hard disk, thus shifting to orbit "2." It then recollides with the disk and shifts to orbit "3." The cyclotron radius is R and the distance between the centers of the cyclotron orbits to the center of the scatterer is Δ. The angles separating two subsequent collisions is 2β.

surface of the scatterer where a new collision occurs, and the particle is shifted to a third cyclotron orbit "3." The direction of **n** in the first collision and **n** of the second collision is shifted by an angle 2β, where β is given by

$$\cos\beta = \frac{\Delta^2 - R^2 + a^2}{2a\Delta} \tag{70}$$

Here R is the cyclotron radius, and a is the radius of the scatterer.

The angle between two subsequent points of collision is 2β as seen from the center of the scatterer. Similarly, the angle between these two points, as seen from the center of the cyclotron orbit one to the other, is 2γ, with

$$\cos\gamma = \frac{R^2 + \Delta^2 - a^2}{2R\Delta} \tag{71}$$

The scattering angle in the second collision remains equal to that of the first, ψ. Note that in the Grad limit, $\beta = (\pi - \psi)/2$ can take any value from 0 to $\pi/2$. On the other hand, $\gamma \to 0$ for *all* collisions in the Grad limit.

Thus, if we set $\mathbf{v} = (v, \varphi_v)$ and $\mathbf{n} = (1, \varphi_n)$ in polar coordinates, the shift operator S_a is, with reference to the figure,

$$S_a(v, \varphi_v) = (v, \varphi_v + \psi - 2\gamma) \xrightarrow{\text{Grad}} (v, \varphi_v + \psi) \tag{72}$$

with γ given by (71). Similarly,

$$S_a(1, \varphi_n) = (1, \varphi_n + 2\beta) \tag{73}$$

where β is given by (70).

In polar coordinates, the generalized Boltzmann equation (1) becomes

$$\frac{D}{Dt} f^G(\mathbf{x}, \varphi_v, t) = \frac{na}{4} \sum_{k=0}^{[t/T_0]} e^{-\nu k T_0} \int_{-\pi}^{\pi} d\psi \sin\left|\frac{\psi}{2}\right| \times [f^G(\mathbf{x}, \varphi_v - (k+1)\psi, t - kT_0) - f^G(\mathbf{x}, \varphi_v - k\psi, t - k) \tag{74}$$

This was the form of the generalized Boltzmann equation used in refs. 4 and 5.

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